J. Math. Anal. Appl. 363 (2010) 679–689

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

The Dirichlet problem of higher order quasilinear elliptic equation ☆

Yongqiang Fu, Mei Yu *

Department of Mathematics, Harbin Institute of Technology, Harbin 150001, China

ARTICLE INFO

Article history:
Received 10 March 2009
Available online 3 October 2009
Submitted by P.G. Lemarie-Rieusset

Keywords:
$L^{p(x)}(\Omega)$ space
$W^{m,p(x)}(\Omega)$ space
Embedding

ABSTRACT

The paper is concerned with the Dirichlet problem of higher order quasilinear elliptic equation:

$$\begin{cases} \sum\limits_{|\alpha|\leqslant m}(-1)^{|\alpha|}D^\alpha A_\alpha\big(x,\xi_m(u)\big)=G+f(x,u), & x\in\Omega,\\ D^\beta u=0, \quad x\in\partial\Omega,\ \forall|\beta|\leqslant m-1, \end{cases}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain, $1<p^-\leqslant p^+<\frac{N}{m}$. G is a bounded linear functional on $W_0^{m,p(x)}(\Omega)$, and $A_\alpha(x,\xi_m(u))$, $f(x,u)$ satisfy Carathéodory condition and some $p(x)$-growth conditions, respectively. We show that there exists at least one non-trivial solution for the above problem in $W_0^{m,p(x)}(\Omega)$.

1. Introduction

After Kováčik and Rákosník first discussed $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$ spaces in [10], a lot of research has been done concerning these kinds of variable exponent spaces, for example, see [4–7] for the properties of such spaces and [3,8,9] for the applications of variable exponent spaces on partial differential equations. Especially in $W^{1,p(x)}(\Omega)$ space, there are a lot of studies for $p(x)$-Laplacian problems, see [8,9]. In recent years, the theory on problems with $p(x)$-growth conditions has important applications in nonlinear elastic mechanics and electrorheological fluids (see [1,2,11,13,15]).

In this paper, we consider the problem

$$\begin{cases} \sum\limits_{|\alpha|\leqslant m}(-1)^{|\alpha|}D^\alpha A_\alpha\big(x,\xi_m(u)\big)=G+f(x,u), & x\in\Omega,\\ D^\beta u=0, \quad x\in\partial\Omega,\ \forall|\beta|\leqslant m-1, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N$ is a bounded domain, $N\geqslant 1$, $p(x)$ is Lipschitz continuous and $1<p^-=\operatorname{ess\,inf}_{x\in\Omega}p(x)\leqslant p^+=\operatorname{ess\,sup}_{x\in\Omega}p(x)<\frac{N}{m}$, G is a bounded linear functional on $W_0^{m,p(x)}(\Omega)$, $\xi_m=\{\xi_\alpha\colon |\alpha|\leqslant m\}$ are the elements in the vector space $\mathbb{R}^m$. We can write ξ_m as $\xi_m=(\eta_{m-1},\zeta_m)$, where $\eta_{m-1}=\{\xi_\alpha\colon |\alpha|\leqslant m-1\}$ and $\zeta_m=\{\xi_\alpha\colon |\alpha|=m\}$. For $u\in W_0^{m,p(x)}(\Omega)$, $\xi_m(u(x))=\{\xi_\alpha(u(x))=D^\alpha u(x)\colon |\alpha|\leqslant m\}$.

When $p(x)$ is a constant function, Shapiro (see [14]) studied the problem above. In this paper, by a special technique, we overcome difficulties caused by $p(x)$-growth conditions.

☆ Supported by Science Research Foundation in Harbin Institute of Technology (HITC200702), and the Natural Science Foundation of Heilongjiang Province (A2007-04) and the Program of Excellent Team in Harbin Institute of Technology.

* Corresponding author.
E-mail address: yumei301796@gmail.com (M. Yu).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2009.10.003

We impose the following conditions on $A_\alpha(x,\xi_m)$:

(A-1) Each $A_\alpha : \Omega \times \mathbb{R}^m \to \mathbb{R}$ satisfies the Carathéodory condition.
(A-2) $|A_\alpha(x,\xi_m)| \leqslant h_0(x) + c_1|\xi_m|^{p(x)-1}$, where $c_1 > 0$ and $0 \leqslant h_0(x) \in L^{p'(x)}(\Omega)$, $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.
(A-3) $\sum_{|\alpha|=m}[A_\alpha(x,\xi_m) - A_\alpha(x,\eta_{m-1},\zeta'_m)](\xi_\alpha - \xi'_\alpha) > 0$ for a.e. $x \in \Omega$, where $\zeta_m \neq \zeta'_m$.
(A-4) $\sum_{|\alpha|\leqslant m} A_\alpha(x,\xi_m)\xi_\alpha \geqslant c_2|\zeta_m|^{p(x)} - H(x)$ for a.e. $x \in \Omega$, where $c_2 > 0$ and $0 \leqslant H(x) \in L^1(\Omega)$.

We define

$$Q(u,v) = \sum_{|\alpha|\leqslant m} \int_\Omega A_\alpha\big(x,\xi_m(u)\big) D^\alpha v\, dx \tag{1.2}$$

on $W_0^{m,p(x)}(\Omega) \times W_0^{m,p(x)}(\Omega)$ and

$$\lambda_1 = \varliminf_{\|u\|_{p(x)}\to\infty} Q(u,u) \Big/ \int_\Omega |u|^{p(x)}\, dx. \tag{1.3}$$

Note that we can apply (A-4) to obtain

$$\frac{Q(u,u)}{\int_\Omega |u|^{p(x)}\,dx} \geqslant \frac{c_2\int_\Omega |\zeta_m(u)|^{p(x)}\,dx}{\int_\Omega |u|^{p(x)}\,dx} - \frac{\int_\Omega H(x)\,dx}{\int_\Omega |u|^{p(x)}\,dx}$$

and

$$\lim_{\|u\|_{p(x)}\to\infty} \frac{\int_\Omega H(x)\,dx}{\int_\Omega |u|^{p(x)}\,dx} = 0.$$

It is immediate that $\lambda_1 \geqslant 0$.

For the function $f(x,t)$, we suppose

(f-1) $f(x,t)$ satisfies the Carathéodory condition.
(f-2) $|f(x,t)| \leqslant h_1(x) + K|t|^{q(x)-1}$ for a.e. $x \in \Omega$ and $t \in \mathbb{R}$, where $K > 0$ and $0 \leqslant h_1(x) \in L^{q'(x)}(\Omega)$, $q : \overline{\Omega} \to \mathbb{R}$ is measurable and satisfies $p(x) \leqslant q(x) \leqslant p^*(x) := \frac{Np(x)}{N-mp(x)}$.
(f-3) $tf(x,t) \leqslant (\lambda_1 - \varepsilon_1)|t|^{p(x)} + h_2(x)|t|$ for a.e. $x \in \Omega$ and $t \in \mathbb{R}$, where $\varepsilon_1 > 0$ and $0 \leqslant h_2(x) \in L^{q'(x)}(\Omega)$.

If $\lambda_1 = \infty$, there exist $M > 0$ and $L > 0$ such that $Q(u,u) \geqslant L\int_\Omega |u|^{p(x)}\,dx$ as $\|u\|_{p(x)} > M$. We replace λ_1 by L in (f-3), then we can obtain the same result.

Our main result is that:

Theorem 1.1. *Under assumptions* (A-1)–(A-4) *and* (f-1)–(f-3)*, there exists* $u^* \in W_0^{m,p(x)}(\Omega)$ *such that for all* $v \in W_0^{m,p(x)}(\Omega)$*,*

$$Q\big(u^*,v\big) = G(v) + \int_\Omega f\big(x,u^*\big) v\, dx. \tag{1.4}$$

This paper is organized as follows: in Section 2 we present some necessary facts; in Section 3, we give the main results.

2. Preliminaries

In this section we first recall some facts on variable exponent spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, where $\Omega \subset \mathbb{R}^N$ is a measurable subset, see [6,7,10] for the details.

Let $\mathbf{P}(\Omega)$ be the set of all Lebesgue measurable functions $p : \Omega \to (1,\infty)$,

$$\|u\|_{p(x)} = \inf\left\{\lambda > 0\colon \int_\Omega \left|\frac{u}{\lambda}\right|^{p(x)} dx \leqslant 1\right\}. \tag{2.1}$$

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is the class of functions u such that $\int_\Omega |u(x)|^{p(x)}\,dx < \infty$. $L^{p(x)}(\Omega)$ is a Banach space endowed with the norm (2.1).

For a given $p(x) \in \mathbf{P}(x)$, we define the conjugate function $p'(x)$ as:

$$p'(x) = \frac{p(x)}{p(x)-1}.$$

Theorem 2.1. *The inequality*

$$\int_{\Omega} \left| f(x) \cdot g(x) \right| dx \leqslant 2 \|f\|_{p(x)} \|g\|_{p'(x)}$$

holds for every $f \in L^{p(x)}(\Omega)$ and $g \in L^{p'(x)}(\Omega)$.

Theorem 2.2. *The dual space to $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$ if and only if $p \in L^{\infty}(\Omega)$. The space $L^{p(x)}(\Omega)$ is reflexive if and only if*

$$1 < p^- \leqslant p^+ < \infty. \tag{2.2}$$

Theorem 2.3. *Suppose that $p(x)$ satisfies* (2.2)*. Let* meas $\Omega < \infty$ *and $p_1(x), p_2(x) \in \mathbf{P}(\Omega)$, then the necessary and sufficient condition for $L^{p_2(x)}(\Omega) \subset L^{p_1(x)}(\Omega)$ is that for almost all $x \in \Omega$ we have $p_1(x) \leqslant p_2(x)$, and in this case the embedding is continuous.*

Definition 2.1. Let $u \in L^{p(x)}(\Omega)$, $D \subset \Omega$ be a measurable subset, and χ_D be the characteristic function of D. If $\lim_{\operatorname{meas} D \to 0} \|u(x)\chi_D(x)\|_{p(x)} = 0$, then we say that u is absolutely continuous with respect to norm $\|\cdot\|_{p(x)}$.

Theorem 2.4. *$u \in L^{p(x)}(\Omega)$ is absolutely continuous with respect to norm $\|\cdot\|_{p(x)}$.*

Theorem 2.5. *Suppose that $p(x)$ satisfies* (2.2) *and $u \in L^{p(x)}(\Omega)$, then*

1. $\|u\|_{p(x)} < 1$ $(= 1; > 1)$ *if and only if* $\rho(u) < 1$ $(= 1; > 1)$.
2. *If $\|u\|_{p(x)} > 1$, then $\|u\|_{p(x)}^{p^-} \leqslant \rho(u) \leqslant \|u\|_{p(x)}^{p^+}$.*
3. *If $\|u\|_{p(x)} < 1$, then $\|u\|_{p(x)}^{p^+} \leqslant \rho(u) \leqslant \|u\|_{p(x)}^{p^-}$.*

Next we assume that $\Omega \subset \mathbb{R}^N$ is a nonempty open set and k is a given natural number.

Given a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_n$, we set $|\alpha| = \alpha_1 + \cdots + \alpha_n$ and $D^{\alpha} = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$, where $D_i = \partial/\partial x_i$ is the generalized derivative operator.

The generalized Sobolev space $W^{k,p(x)}(\Omega)$ is the class of functions f on Ω such that $D^{\alpha} f \in L^{p(x)}(\Omega)$ for every multi-index α with $|\alpha| \leqslant k$. $W^{k,p(x)}(\Omega)$ is endowed with the norm

$$\|f\|_{k,p(x)} = \sum_{|\alpha| \leqslant k} \left\| D^{\alpha} f \right\|_{p(x)}. \tag{2.3}$$

By $W_0^{k,p(x)}(\Omega)$ we denote the subspace of $W^{k,p(x)}(\Omega)$ which is the closure of $C_0^{\infty}(\Omega)$ with respect to the norm (2.3).

Theorem 2.6. *The spaces $W^{k,p(x)}(\Omega)$ and $W_0^{k,p(x)}(\Omega)$ are separable reflexive Banach spaces, if $p(x)$ satisfies* (2.2)*.*

We denote the dual space of $W_0^{k,p(x)}(\Omega)$ by $W^{-k,p'(x)}(\Omega)$, then we have

Theorem 2.7. *For every $G \in W^{-k,p'(x)}(\Omega)$ there exists a unique system of functions $\{g_{\alpha} \in L^{p'(x)}(\Omega)\colon |\alpha| \leqslant k\}$ such that*

$$G(f) = \sum_{|\alpha| \leqslant k} \int_{\Omega} D^{\alpha} f(x) g_{\alpha}(x)\, dx, \quad \forall f \in W_0^{k,p(x)}(\Omega).$$

The norm of $W^{-k,p'(x)}(\Omega)$ is defined as

$$\|G\|_{-k,p'} = \sup\left\{ \frac{|G(f)|}{\|f\|_{k,p(x)}}\colon f \in W_0^{k,p(x)}(\Omega)\backslash\{0\} \right\}.$$

Theorem 2.8. *Let Ω be a domain in $\mathbb{R}^n$ with cone property. If $p : \overline{\Omega} \to R$ is Lipschitz continuous and $1 < p^- \leqslant p^+ < \frac{N}{k}$, $q(x) : \overline{\Omega} \to \mathbb{R}$ is measurable and satisfies $p(x) \leqslant q(x) \leqslant p^*(x) := \frac{Np(x)}{N-kp(x)}$ for a.e. $x \in \overline{\Omega}$, then there is a continuous embedding $W^{k,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.*

Theorem 2.9. *Let Ω be a bounded domain. If $p(x) \in L^{\infty}(\Omega)$ and $u \in W_0^{1,p(x)}(\Omega)$, then*

$$\|u\|_{p(x)} \leqslant C \|\nabla u\|_{p(x)},$$

where C is a constant dependent on Ω.

3. The Dirichlet problem

In this section we will give the proof of Theorem 1.1. First we give some lemmas.

Lemma 3.1. *Let Ω be a bounded open set, then*

(a) $\exists\{\phi_n\}_{n=1}^{\infty}$ *which is a complete orthonormal system in* $L^2(\Omega)$ *with* $\phi_n \in W_0^{m,2}(\Omega) \cap W_0^{m,p(x)}(\Omega)$, $\forall n$;
(b) *Let S_J be the subspace in $W_0^{m,p(x)}(\Omega)$ spanned by $\{\phi_1, \phi_2, \dots, \phi_J\}$. Then given $v \in W_0^{m,p(x)}(\Omega)$, $\exists\{v_J\}_{J=1}^{\infty}$ with $v_J \in S_J$ such that* $\lim_{J\to\infty}\|v - v_J\|_{m,p(x)} = 0$.

Proof. When $p(x) \equiv p > 1$ is a constant function, Lemma 3.1 first is proved in the case $2 \leqslant p < \infty$, then the fact that $W_0^{m,2}(\Omega)$ is dense in $W_0^{m,p}(\Omega)$ is used to show that Lemma 3.1 holds for $1 < p < 2$. So $\exists\{\phi_n\}_{n=1}^{\infty} \subset W_0^{m,2}(\Omega)\bigcap W_0^{m,p}(\Omega)$ such that (a) and (b) hold for $1 < p(x) \equiv p < \infty$ (see [14, Appendix]). By Theorem 2.3, we have the embedding $W^{m,p^+}(\Omega) \hookrightarrow W^{m,p(x)}(\Omega)$, so $\exists\{\phi_n\}_{n=1}^{\infty}$ such that $\{\phi_n\}_{n=1}^{\infty} \subset W_0^{m,2}(\Omega)\bigcap W_0^{m,p^+}(\Omega) \subset W_0^{m,2}(\Omega)\bigcap W_0^{m,p(x)}(\Omega)$ and $\{\phi_n\}_{n=1}^{\infty}$ is a complete orthonormal system in $L^2(\Omega)$. Then $\forall v \in W_0^{m,p(x)}(\Omega)$, $\exists\{f_J\} \subset C_0^{\infty}(\Omega)$ such that $\forall\varepsilon > 0$, $\exists N$, $\|v - f_J\|_{m,p(x)} < \varepsilon$ whenever $J > N$. Since $f_J \in W_0^{m,p^+}(\Omega)$, $\exists M_J > 0$ such that $\|\sum_{j=1}^{M_J} c_{J_j}\phi_j - f_J\|_{m,p^+} < \varepsilon$. By the embedding $W^{m,p^+}(\Omega) \hookrightarrow W^{m,p(x)}(\Omega)$, we have $\|u\|_{m,p(x)} \leqslant C\|u\|_{m,p^+}$. Furthermore when J is sufficiently large, we have

$$\begin{aligned}\left\|v - \sum_{j=1}^{M_J} c_{J_j}\phi_j\right\|_{m,p(x)} &\leqslant \|v - f_J\|_{m,p(x)} + \left\|f_J - \sum_{j=1}^{M_J} c_{J_j}\phi_j\right\|_{m,p(x)}\\ &\leqslant \varepsilon + C\left\|f_J - \sum_{j=1}^{M_J} c_{J_j}\phi_j\right\|_{m,p^+}\\ &\leqslant \varepsilon + C\varepsilon.\end{aligned}$$

Take $v_J = \sum_{j=1}^{M_J} c_{J_j}\phi_j$, then $\|v - v_J\| < \varepsilon$ when J is sufficiently large. □

Lemma 3.2. *Let $J \geqslant 1$ be a given positive integer. Assume the hypotheses of Theorem* 1.1. *Then there exists $u_J \in S_J$ such that for all $v \in S_J$,*

$$Q(u_J, v) = G(v) + \int_{\Omega} f(x, u_J)v\,dx,$$

where S_J is the subspace in Lemma 3.1(b).

Proof. Since $G \in W^{-m,p'(x)}(\Omega)$, $\exists C > 0$ such that for every $v \in W_0^{m,p(x)}(\Omega)$,

$$\left|G(v)\right| \leqslant C\|v\|_{m,p(x)}. \tag{3.1}$$

By Theorem 2.8 and (f-2), $\exists C > 0$ such that for every $v \in W_0^{m,p(x)}(\Omega)$,

$$\|v\|_{q(x)} \leqslant C\|v\|_{m,p(x)}. \tag{3.2}$$

Since the dimension of S_J is finite, $\exists C > 0$ such that for every $\alpha \in \mathbb{R}^J$,

$$C\left\|\sum_{j=1}^{J}\alpha_j\phi_j\right\|_{p(x)} \geqslant \left\|\sum_{j=1}^{J}\alpha_j\phi_j\right\|_{m,p(x)}. \tag{3.3}$$

Let $\beta = (\beta_1, \beta_2, \dots, \beta_J) \in \mathbb{R}^J$,

$$F_k(\beta) = Q\left(\sum_{j=1}^{J}\beta_j\phi_j, \phi_k\right) - G(\phi_k) - \int_{\Omega} f\left(x, \sum_{j=1}^{J}\beta_j\phi_j\right)\phi_k\,dx,$$

$k = 1, \dots, J$ and $F(\beta) = (F_1(\beta), \dots, F_J(\beta))$. Since the dimension of S_J is finite and $\|\sum_{j=1}^{J}\beta_j\phi_j\|_{L^2} \to \infty$ as $|\beta| \to \infty$, $\|\sum_{j=1}^{J}\beta_j\phi_j\|_{p(x)} \to \infty$ as $|\beta| \to \infty$. By (1.3), $\exists\gamma_0 > 0$ such that whenever $|\beta| \geqslant \gamma_0$,

$$Q\left(\sum_{j=1}^{J}\beta_j\phi_j, \sum_{j=1}^{J}\beta_j\phi_j\right) \geqslant \left(\lambda_1 - \frac{\varepsilon_1}{2}\right)\int_{\Omega}\left|\sum_{j=1}^{J}\beta_j\phi_j\right|^{p(x)} dx.$$

Then

$$F(\beta)\beta \geqslant \left(\lambda_1 - \frac{\varepsilon_1}{2}\right)\int_\Omega \left|\sum_{j=1}^J \beta_j\phi_j\right|^{p(x)} dx - G\left(\sum_{j=1}^J \beta_j\phi_j\right) - \int_\Omega f\left(x, \sum_{j=1}^J \beta_j\phi_j\right)\left(\sum_{k=1}^J \beta_k\phi_k\right) dx.$$

From the results above, (f-3), (3.1)–(3.3), we obtain

$$\begin{aligned} F(\beta)\beta &\geqslant \frac{\varepsilon_1}{2}\int_\Omega \left|\sum_{j=1}^J \beta_j\phi_j\right|^{p(x)} dx - C\left\|\sum_{j=1}^J \beta_j\phi_j\right\|_{m,p(x)} - \int_\Omega h_2(x)\left|\sum_{j=1}^J \beta_j\phi_j\right| dx \\ &\geqslant \frac{\varepsilon_1}{2}\left\|\sum_{j=1}^J \beta_j\phi_j\right\|_{p(x)}^{p^-} - C\left(\left\|\sum_{j=1}^J \beta_j\phi_j\right\|_{m,p(x)} - \left\|h_2(x)\right\|_{q'(x)}\cdot\left\|\sum_{j=1}^J \beta_j\phi_j\right\|_{m,p(x)}\right) \\ &\geqslant \frac{\varepsilon_1}{2}\left\|\sum_{j=1}^J \beta_j\phi_j\right\|_{p(x)}^{p^-} - C\left\|\sum_{j=1}^J \beta_j\phi_j\right\|_{p(x)}, \end{aligned}$$

so $\exists\gamma_1$ such that $F(\beta)\beta > 0$ for $|\beta| \geqslant \gamma_1 > \gamma_0$.

It follows from (A-1), (A-2), (1.2), (f-2) and the definition of $F_k(\beta)$ that $F_k(\beta) \in C(R^J)$ for $k = 1,\dots,J$. Hence, by [12], it is immediate that $\exists l = (l_1,\dots,l_J)$ with $|l| < \gamma_1$ such that $F_k(l) = 0$ for $k = 1,\dots,J$. Setting $u_J = \sum_{j=1}^J l_j\phi_j$, we obtain $Q(u_J,\phi_k) = G(\phi_k) + \int_\Omega f(x,u_J)\phi_k\,dx$ for $k = 1,\dots,J$. Furthermore for any $v = \sum_{k=1}^J \beta_k\phi_k \in S_J$, $Q(u_J,v) = G(v) + \int_\Omega f(x,u_J)v\,dx$. $\square$

Lemma 3.3. *Under the assumptions of Lemma 3.2, there is a sequence $\{u_J\}_{J=1}^\infty$ with $u_J \in S_J$ such that the conclusion of Lemma 3.2 holds, then $\exists C > 0$ such that for any J, $\|u_J\|_{p(x)} \leqslant C$.*

Proof. By Lemma 3.2, it is easy to get a sequence $\{u_J\}_{J=1}^\infty$ with $u_J \in S_J$ such that the conclusion of Lemma 3.2 holds. Suppose that C does not exist and $\lim_{J\to\infty}\|u_J\|_{p(x)} = \infty$. By (1.3), (3.1), (3.2) and (f-3), we get

$$\begin{aligned} \left(\lambda_1 - \frac{\varepsilon_1}{2}\right)\int_\Omega |u_J|^{p(x)}\,dx \leqslant Q(u_J,u_J) &\leqslant C\|u_J\|_{m,p(x)} + (\lambda_1-\varepsilon_1)\int_\Omega |u_J|^{p(x)}\,dx + \int_\Omega h_2(x)|u_J|\,dx \\ &\leqslant C\|u_J\|_{m,p(x)} + (\lambda_1-\varepsilon_1)\int_\Omega |u_J|^{p(x)}\,dx + 2\left\|h_2(x)\right\|_{q'(x)}\cdot\|u_J\|_{m,p(x)} \end{aligned}$$

for $J \geqslant J_0 > 0$. So $\exists C > 0$ such that for any J, $\int_\Omega |u_J|^{p(x)}\,dx \leqslant C\|u_J\|_{m,p(x)}$. By (A-4)

$$C\int_\Omega \left|\zeta_m(u_J)\right|^{p(x)} dx \leqslant \int_\Omega H(x)\,dx + G(u_J) + \int_\Omega f(x,u_J)u_J\,dx,$$

and by (3.2)

$$\int_\Omega f(x,u_J)u_J\,dx \leqslant |\lambda_1-\varepsilon_1|\int_\Omega |u_J|^{p(x)}\,dx + \int_\Omega h(x)|u_J|\,dx \leqslant C\left(|\lambda_1-\varepsilon_1|\|u_J\|_{m,p(x)} + \left\|h(x)\right\|_{q'(x)}\|u_J\|_{m,p(x)}\right),$$

so by (3.1) it is immediate that

$$C\int_\Omega \left|\zeta_m(u_J)\right|^{p(x)} dx \leqslant \int_\Omega H(x)\,dx + \|u_J\|_{m,p(x)}.$$

By Theorems 2.5 and 2.9 we get $\|u_J\|_{m,p(x)}^{p^-} \leqslant C\sum_{|\alpha|\leqslant m}\|D^\alpha u_J\|_{p(x)}^{p^-}$ and $\sum_{|\alpha|=m-1}\|D^\alpha u_J\|_{p(x)}^{p^-} \leqslant C\|\zeta_m(u_J)\|_{p(x)}^{p^-}$. Then we have

$$\int_\Omega \left|\zeta_m(u_J)\right|^{p(x)} dx \geqslant \left\|\zeta_m(u_J)\right\|_{p(x)}^{p^-} \geqslant C\|u_J\|_{m,p(x)}^{p^-},$$

because $\|\zeta_m(u_J)\|_{p(x)} > 1$ when J is sufficiently large. Furthermore

$$C\|u_J\|_{m,p(x)}^{p^-} \leqslant \int_\Omega H(x)\,dx + \|u_J\|_{m,p(x)}.$$

Since $p^- > 1$, we obtain that $\{\|u_J\|_{m,p(x)}\}_{J=1}^\infty$ is bounded, so $\{\|u_J\|_{p(x)}\}_{J=1}^\infty$ is also bounded. We obtain a contradiction. $\square$

Lemma 3.4. *Under the assumptions of Lemma* 3.3, *there exist a subsequence* (*still denote it by* $\{u_J\}_{J=1}^{\infty}$) *and* $u^* \in W_0^{m,p(x)}(\Omega)$ *such that*

$$\lim_{J\to\infty}\sum_{|\alpha|=m}\left[A_\alpha\left(x,\xi_m(u_J)\right)-A_\alpha\left(x,\eta_{m-1}(u_J),\zeta_m\left(u^*\right)\right)\right]\left[D^\alpha u_J-D^\alpha u^*\right]=0$$

for a.e. $x\in\Omega$.

Proof. *Step* 1. By Lemma 3.3 and Theorem 2.5, $\exists C>0$ such that $\int_\Omega |u_J|^{p(x)}\,dx<C$, then by (3.2) we get

$$\int_\Omega f(x,u_J)u_J\,dx\leqslant|\lambda_1-\varepsilon_1|\int_\Omega|u_J|^{p(x)}\,dx+\int_\Omega h_2(x)|u_J|\,dx\leqslant C\left(|\lambda_1-\varepsilon_1|+\left\|h_2(x)\right\|_{q'(x)}\cdot\|u_J\|_{m,p(x)}\right).$$

Then similar to the proof of Lemma 3.3

$$C\|u_J\|_{m,p(x)}^{p^-}\leqslant C\left(\|u_J\|_{m,p(x)}+\int_\Omega H(x)\,dx+|\lambda_1-\varepsilon_2|+\left\|h_2(x)\right\|_{q'(x)}\cdot\|u_J\|_{m,p(x)}\right).$$

We obtain that $\{\|u_J\|_{m,p(x)}\}_{J=1}^{\infty}$ is bounded, i.e. $\exists K_6>0$ such that for any J,

$$\|u_J\|_{m,p(x)}\leqslant K_6. \tag{3.4}$$

As $W_0^{m,p(x)}(\Omega)$ is a separable reflexive Banach space, there exist a subsequence (still denote it by $\{u_J\}_{J=1}^{\infty}$) and $u^*\in W_0^{m,p(x)}(\Omega)$ such that

$$\lim_{J\to\infty}\left\|D^\alpha u_J-D^\alpha u^*\right\|_{p(x)}=0\quad\text{for }|\alpha|\leqslant m-1, \tag{3.5}$$

and

$$\lim_{J\to\infty}\int_\Omega D^\alpha u_J\,w\,dx=\int_\Omega D^\alpha u^*\,w\,dx,\quad\forall w\in L^{p'(x)}(\Omega),\ \text{for }|\alpha|=m, \tag{3.6}$$

and

$$\lim_{J\to\infty}G(u_J)=G\left(u^*\right), \tag{3.7}$$

and

$$\lim_{J\to\infty}\eta_{m-1}\left(u_J(x)\right)=\eta_{m-1}\left(u^*(x)\right)\quad\text{for a.e. }x\in\Omega. \tag{3.8}$$

We define

$$V_\alpha=A_\alpha\left(x,\eta_{m-1}(u_J),\zeta_m\left(u^*\right)\right)-A_\alpha\left(x,\xi_m\left(u^*\right)\right).$$

Note that for $|\alpha|=m$,

$$\begin{aligned}&\int_\Omega A_\alpha\left(x,\eta_{m-1}(u_J),\zeta_m\left(u^*\right)\right)\left[D^\alpha u_J(x)-D^\alpha u^*(x)\right]dx\\&\quad=\int_\Omega\left\{V_\alpha\left[D^\alpha u_J(x)-D^\alpha u^*(x)\right]+A_\alpha\left(x,\xi_m\left(u^*\right)\right)\left[D^\alpha u_J-D^\alpha u^*\right]\right\}dx.\end{aligned}$$

We will prove

$$\lim_{J\to\infty}\int_\Omega\sum_{|\alpha|=m}A_\alpha\left(x,\eta_{m-1}(u_J),\zeta_m\left(u^*\right)\right)\left[D^\alpha u_J-D^\alpha u^*\right]dx=0. \tag{3.9}$$

By (A-2)

$$\int_\Omega \left|A_\alpha\big(x,\xi_m(u^*)\big)\right|^{p'(x)}dx \leqslant \int_\Omega \left|h_0(x)+c_1\left|\xi_m(u^*)\right|^{p(x)-1}\right|^{p'(x)}dx$$
$$\leqslant C\bigg(\int_\Omega \left|h_0(x)\right|^{p'(x)}dx+\int_\Omega \left|\xi_m(u^*)\right|^{p(x)}dx\bigg)$$
$$<\infty,$$

then we have

$$A_\alpha\big(x,\xi_m(u^*)\big)\in L^{p'(x)}\quad \text{for } |\alpha|=m.$$

By (3.6)

$$\lim_{J\to\infty}\int_\Omega A_\alpha\big(x,\xi_m(u^*)\big)\big[D^\alpha u_J-D^\alpha u^*\big]\,dx=0\quad \text{for } |\alpha|=m.$$

Furthermore by Theorem 2.4, $\forall\varepsilon>0$, $\exists\delta>0$ such that whenever $\operatorname{meas} E<\delta$ we have $\int_E |A_\alpha(x,\xi_m(u^*))|^{p'(x)}\,dx<\varepsilon$, $\int_E |h_0(x)|^{p'(x)}\,dx<\varepsilon$ and $\int_E |D^\alpha u^*|^{p(x)}\,dx<\varepsilon$. From (3.5) and Theorem 2.5

$$\lim_{J\to\infty}\int_\Omega \left|D^\alpha u_J-D^\alpha u^*\right|^{p(x)}dx=0$$

for $|\alpha|\leqslant m-1$ and further $\exists J_0$ such that whenever $J\geqslant J_0$, $\int_\Omega |D^\alpha u_J-D^\alpha u^*|^{p(x)}\,dx<\varepsilon$ and

$$\int_E \left|D^\alpha u_J\right|^{p(x)}dx\leqslant 2^{p^+}\bigg(\int_E \left|D^\alpha u_J-D^\alpha u^*\right|^{p(x)}dx+\int_E \left|D^\alpha u^*\right|^{p(x)}dx\bigg)\leqslant \big(2^{p^+}+1\big)\varepsilon.$$

By (A-2) we have

$$\int_E |V_\alpha|^{p'(x)}\,dx\leqslant C\int_E \Big(\left|A_\alpha\big(x,\eta_{m-1}(u_J),\zeta_m(u^*)\big)\right|^{p'(x)}+\left|A_\alpha\big(x,\xi_m(u^*)\big)\right|^{p'(x)}\Big)\,dx$$
$$\leqslant C\bigg(\int_E \left|h_0(x)\right|^{p'(x)}dx+\int_E \left|\big(\eta_{m-1}(u_J),\zeta_m(u^*)\big)\right|^{p(x)}dx\bigg)+\varepsilon$$
$$\leqslant C\bigg(\varepsilon+\int_E \left|\eta_{m-1}(u_J)\right|^{p(x)}dx+\int_E \left|\zeta_m(u^*)\right|^{p(x)}dx\bigg)$$
$$\leqslant C\varepsilon.$$

With (3.8) and (A-1), it is immediate that $\lim_{J\to\infty}|V_\alpha|^{p'(x)}=0$ for $|\alpha|=m$. Consequently from Vitali's Theorem $\lim_{J\to\infty}\int_\Omega |V_\alpha|^{p'(x)}\,dx=0$ for $|\alpha|=m$. By Theorem 2.1

$$\int_\Omega V_\alpha\big[D^\alpha u_J(x)-D^\alpha u^*(x)\big]\,dx\leqslant 2\|V_\alpha\|_{p'(x)}\left\|D^\alpha u_J-D^\alpha u^*\right\|_{p(x)}.$$

Then $\lim_{J\to\infty}\int_\Omega V_\alpha[D^\alpha u_J-D^\alpha u^*]\,dx=0$ for $|\alpha|=m$. So we obtain (3.9).

Step 2. We will prove

$$\varlimsup_{J\to\infty}\int_\Omega \sum_{|\alpha|=m}A_\alpha\big(x,\xi_m(u_J)\big)\big[D^\alpha u_J-D^\alpha u^*\big]\,dx\leqslant 0.$$

By Lemma 3.1(b) and $u^*\in W_0^{m,p(x)}(\Omega)$, $\exists\{u_J^*\}_{J=1}^\infty$ with $u_J^*\in S_J$ such that

$$\lim_{J\to\infty}\left\|u_J^*-u^*\right\|_{m,p(x)}=0. \tag{3.10}$$

We observe that

$$Q\left(u_J,u_J-u_J^*\right)=G\big(u_J-u_J^*\big)+\int_\Omega f(x,u_J)\big(u_J-u_J^*\big)\,dx.$$

By (3.1), (3.7) and (3.10), it is immediate that

$$\left|G(u_J)-G\left(u_J^*\right)\right| \leqslant \left|G\left(u^*-u_J^*\right)\right| + \left|G(u_J)-G\left(u^*\right)\right| \leqslant C\left\|u^*-u_J^*\right\|_{m,p(x)} + \left|G(u_J)-G\left(u^*\right)\right|.$$

So $\lim_{J\to\infty}|G(u_J)-G(u_J^*)|=0$. It follows from (3.2) and (3.4) that $\exists C>0$ such that $\|u_J\|_{q(x)} \leqslant C$ for $J=1,2,\ldots$. By (f-2), $\exists C>0$ such that for any J, $\int_\Omega |f(x,u_J)|^{q'(x)}\,dx \leqslant C$. Therefore by Theorem 2.4, $\forall\varepsilon>0$, $\exists\delta>0$ such that whenever $\operatorname{meas} E<\delta$,

$$\int_E \left|f(x,u_J)u^*\right| \leqslant 2\left\|f(x,u_J)\right\|_{q'(x)}\left\|u^*\right\|_{q(x),E}\,dx \leqslant C\varepsilon.$$

We obtain

$$\int_E \left|f(x,u_J)u^*\right|dx \leqslant \varepsilon \quad \text{for } J=1,2,\ldots. \tag{3.11}$$

From (3.5) we have $\lim_{J\to\infty}\int_\Omega |u_J-u^*|^{p(x)}\,dx=0$. Then $\exists J_0$ such that whenever $J\geqslant J_0$, $\int_\Omega |u_J-u^*|^{p(x)}\,dx<\varepsilon$. By Theorem 2.4, $\int_E |u^*|^{p(x)}\,dx<\varepsilon$ and further we have

$$\int_E |u_J|^{p(x)}\,dx < C\left(\int_E \left|u_J-u^*\right|^{p(x)}dx + \int_E \left|u^*\right|^{p(x)}dx\right) < C\varepsilon$$

for $J\geqslant J_0$. With $\int_E |h_2(x)|^{q'(x)}\,dx<\varepsilon$ and (f-3), we have

$$\int_E f(x,u_J)u_J\,dx \leqslant (\lambda_1+\varepsilon_1)\int_E |u_J|^{p(x)}\,dx + \int_E \left|h_2(x)\right|\cdot|u_J|\,dx \leqslant C\varepsilon \tag{3.12}$$

for $J=1,2,\ldots$. By (3.2) and (3.10), $\lim_{J\to\infty}\|u_J^*-u^*\|_{q(x)}=0$. By Theorem 2.1, we have $\lim_{J\to\infty}\int_\Omega f(x,u_J)(u_J^*-u^*)\,dx=0$. By (f-1) and (3.8), we have $\lim_{J\to\infty} f(x,u_J)=f(x,u^*)$ a.e. on Ω, then $\lim_{J\to\infty} f(x,u_J)(u_J-u^*)=0$ a.e. on Ω. Using Egoroff's theorem we see that $\exists E$ with $\operatorname{meas} E<\delta$ such that $\lim_{J\to\infty} f(x,u_J)(u_J-u^*)=0$ uniformly in $\Omega\backslash E$. Consequently $\exists J_0$ such that for any $J\geqslant J_0$, $|f(x,u_J)(u_J-u^*)|<\varepsilon/\operatorname{meas}\Omega$ for $x\in\Omega\backslash E$, and with (3.11), (3.12), we have for $J\geqslant J_0$,

$$\int_\Omega f(x,u_J)\left(u_J-u^*\right)dx \leqslant \int_E \left|f(x,u_J)u_J\right|dx + \int_E \left|f(x,u_J)u^*\right|dx + \int_{\Omega\backslash E} \left|f(x,u_J)\left(u_J-u^*\right)\right|dx \leqslant 3\varepsilon.$$

Since ε is an arbitrary positive number, we get

$$\varlimsup_{J\to\infty}\int_\Omega f(x,u_J)\left(u_J-u^*\right)dx \leqslant 0.$$

Furthermore we get $\varlimsup_{J\to\infty}\int_\Omega f(x,u_J)(u_J-u_J^*)\,dx \leqslant 0$. From the results above, it is immediate that $\varlimsup_{J\to\infty} Q(u_J, u_J-u_J^*) \leqslant 0$ or

$$\varlimsup_{J\to\infty}\int_\Omega \sum_{|\alpha|\leqslant m} A_\alpha\left(x,\xi_m(u_J)\right)\left[D^\alpha u_J - D^\alpha u_J^*\right]dx \leqslant 0.$$

By (A-2) and (3.4), $\exists C>0$ such that

$$\int_\Omega \left|A_\alpha\left(x,\xi_m(u_J)\right)\right|^{p'(x)}dx \leqslant C \quad \text{for } |\alpha|\leqslant m,\ J=1,2,\ldots. \tag{3.13}$$

From (3.10), we get $\lim_{J\to\infty}\int_\Omega A_\alpha(x,\xi_m(u_J))[D^\alpha u_J^*-D^\alpha u^*]\,dx=0$ for $|\alpha|\leqslant m$. Hence $\varlimsup_{J\to\infty}\int_\Omega \sum_{|\alpha|\leqslant m} A_\alpha(x,\xi_m(u_J))\cdot[D^\alpha u_J-D^\alpha u^*]\,dx\leqslant 0$. Note that

$$\int_\Omega A_\alpha\left(x,\xi_m(u_J)\right)\left[D^\alpha u_J - D^\alpha u^*\right]dx \leqslant 2\left\|A_\alpha\left(x,\xi_m(u_J)\right)\right\|_{p'(x)}\left\|D^\alpha u_J - D^\alpha u^*\right\|_{p(x)}$$

for $|\alpha|\leqslant m-1$ and by (3.5) we get $\lim_{J\to\infty}\int_\Omega \sum_{|\alpha|\leqslant m-1} A_\alpha(x,\xi_m(u_J))[D^\alpha u_J-D^\alpha u^*]\,dx=0$. So

$$\varlimsup_{J\to\infty}\int_\Omega \sum_{|\alpha|=m} A_\alpha\left(x,\xi_m(u_J)\right)\left[D^\alpha u_J - D^\alpha u^*\right]dx \leqslant 0. \tag{3.14}$$

Step 3. We define

$$U_\alpha(x) = A_\alpha\big(x, \xi_m(u_J)\big) - A_\alpha\big(x, \eta_{m-1}(u_J), \zeta_m(u^*)\big).$$

We observe that from (3.9) and (3.14)

$$\overline{\lim_{J\to\infty}} \int\limits_\Omega \sum_{|\alpha|=m} U_\alpha(x)\cdot\big[D^\alpha u_J - D^\alpha u^*\big]\,dx \leqslant 0.$$

By (A-3), we get $\lim_{J\to\infty}\int_\Omega \sum_{|\alpha|=m} U_\alpha(x)\cdot[D^\alpha u_J - D^\alpha u^*]\,dx = 0$. So the sequence $\{\sum_{|\alpha|=m} U_\alpha(x)\cdot[D^\alpha u_J - D^\alpha u^*]\}_{J=1}^\infty$ converges in L^1-norm to zero, and further there exists a subsequence (still denote it by $\{u_J\}_{J=1}^\infty$) satisfying $\lim_{J\to\infty}\sum_{|\alpha|=m} U_\alpha(x)\cdot[D^\alpha u_J - D^\alpha u^*] = 0$ for a.e. $x\in\Omega$. □

Lemma 3.5. *Assume that the sequence $\{u_J\}_{J=1}^\infty$ satisfies Lemma* 3.4*, then $\{|\zeta_m(u_J)|\}_{J=1}^\infty$ is pointwise bounded for a.e. $x\in\Omega$, i.e. $\exists$ constant K_x such that $|\zeta_m(u_J)|\leqslant K_x$ for $J=1,\dots$ and a.e. $x\in\Omega$.*

Proof. Let $\Omega_1\subset\Omega$ with $\operatorname{meas}\Omega_1 = \operatorname{meas}\Omega$ be the set such that (3.8) and Lemma 3.4 hold, where $\xi_m(u_J(x))$, $\xi_m(u^*(x))$, $A_\alpha(x, \eta_{m-1}(u_J(x)), \zeta_m(u^*(x)))$, $h_0(x)$ and $H(x)$ are finite-valued for $x\in\Omega_1$, $|\alpha|\leqslant m$ and $J = 1, 2, \dots$. It is sufficient to show that

$$\big\{\big|\zeta_m(u_J)\big|\big\}_{J=1}^\infty \text{ is pointwise bounded for } x\in\Omega_1. \tag{3.15}$$

Suppose to the contrary that there exist a point $x_0\in\Omega_1$ and a subsequence (still denote it by $\{|\zeta_m(u_J)|\}_{J=1}^\infty$) such that $\lim_{J\to\infty}|\zeta_m(u_J(x_0))| = \infty$. Let $0<\varepsilon<1$ and $p(x_0)-\varepsilon>1$. It follows from (A-2) that

$$\frac{A_\alpha(x_0, \xi_m(u_J))D^\alpha u_J}{|\zeta_m(u_J)|^{p(x_0)-\varepsilon}} \leqslant \frac{h_0(x_0)D^\alpha u_J + c_1|\xi_m(u_J)|^{p(x_0)-1}D^\alpha u_J}{|\zeta_m(u_J)|^{p(x_0)-\varepsilon}}$$

for $|\alpha|\leqslant m-1$. Since $p(x_0)-\varepsilon-(p(x_0)-1)>0$, by (3.8) we have

$$\lim_{J\to\infty}\frac{\sum_{|\alpha|\leqslant m-1} A_\alpha(x_0, \xi_m(u_J))D^\alpha u_J}{|\zeta_m(u_J)|^{p(x_0)-\varepsilon}} = 0.$$

For fixed J we have

$$A_\alpha\big(x, \xi_m(u_J)\big)D^\alpha u_J = A_\alpha\big(x, \xi_m(u_J)\big)D^\alpha u^* + A_\alpha\big(x, \eta_{m-1}(u_J), \zeta_m(u^*)\big)\big[D^\alpha u_J - D^\alpha u^*\big] + U_\alpha(x)\big[D^\alpha u_J - D^\alpha u^*\big].$$

By (A-2)

$$\frac{A_\alpha(x_0, \xi_m(u_J))D^\alpha u^*}{|\zeta_m(u_J)|^{p(x_0)-\varepsilon}} \leqslant \frac{h_0(x_0)D^\alpha u^* + c_1|\xi_m(u_J)|^{p(x_0)-1}D^\alpha u^*}{|\zeta_m(u_J)|^{p(x_0)-\varepsilon}},$$

and further

$$\lim_{J\to\infty}\frac{A_\alpha(x_0, \xi_m(u_J))D^\alpha u^*}{|\zeta_m(u_J)|^{p(x_0)-\varepsilon}} = 0$$

for $|\alpha| = m$. It follows from (3.8) that

$$\lim_{J\to\infty}\frac{A_\alpha(x_0, \eta_{m-1}(u_J), \zeta_m(u^*))(D^\alpha u_J - D^\alpha u^*)}{|\zeta_m(u_J)|^{p(x_0)-\varepsilon}} = 0$$

for $|\alpha| = m$. Furthermore by Lemma 3.4

$$\lim_{J\to\infty}\frac{\sum_{|\alpha|=m} U_\alpha(x_0)\cdot[D^\alpha u_J - D^\alpha u^*]}{|\zeta_m(u_J)|^{p(x_0)-\varepsilon}} = 0.$$

So we have

$$\lim_{J\to\infty}\frac{\sum_{|\alpha|=m} A_\alpha(x_0, \xi_m(u_J))D^\alpha u_J}{|\zeta_m(u_J)|^{p(x_0)-\varepsilon}} = 0.$$

By (A-4)

$$\frac{c_2|\zeta_m(u_J)|^{p(x_0)}}{|\zeta_m(u_J)|^{p(x_0)-\varepsilon}} \leqslant \frac{\sum_{|\alpha|\leqslant m} A_\alpha(x_0, \xi_m(u_J))D^\alpha u_J + H(x_0)}{|\zeta_m(u_J)|^{p(x_0)-\varepsilon}},$$

and moreover $\lim_{J\to\infty}|\zeta_m(u_J(x_0))|^\varepsilon = 0$. We get a contradiction, so (3.15) holds, furthermore Lemma 3.5 holds. □

Proof of Theorem 1.1. Let $\{u_J\}_{J=1}^{\infty}$ be the sequence in Lemmas 3.4 and 3.5. Let Ω_2 be the subset of Ω such that $\operatorname{meas}(\Omega\backslash\Omega_2)=0$ and moreover Lemmas 3.4 and 3.5 hold for $x\in\Omega_2$. We will show that there exists a subsequence (still denote it by $\{u_J\}_{J=1}^{\infty}$) such that

$$\lim_{J\to\infty}\zeta_m\big(u_J(x)\big)=\zeta_m\big(u^*(x)\big)\quad\text{for a.e. } x\in\Omega. \tag{3.16}$$

Suppose that there exists $x_0\in\Omega_2$ such that (3.16) does not hold, then there exist a subsequence (still denote it by $\{|\zeta_m(u_J)|\}_{J=1}^{\infty}$) and $\zeta_m^*(u^*(x_0))$ with $\zeta_m^*(u^*(x_0))\neq\zeta_m(u^*(x_0))$ such that $\lim_{J\to\infty}\zeta_m(u_J(x_0))=\zeta_m^*(u^*(x_0))$.

We define

$$W_\alpha=A_\alpha\big(x_0,\eta_{m-1}\big(u^*\big),\zeta_m^*\big(u^*\big)\big)-A_\alpha\big(x_0,\xi_m\big(u^*\big)\big).$$

From (3.8) we have

$$\lim_{J\to\infty}\sum_{|\alpha|=m}U_\alpha(x_0)\cdot\big[\zeta_m(u_J)-\zeta_m\big(u^*\big)\big]=\sum_{|\alpha|=m}W_\alpha\big[\zeta_m^*\big(u^*\big)-\zeta_m\big(u^*\big)\big].$$

By (A-3), it is immediate that $\sum_{|\alpha|=m}W_\alpha[\zeta_m^*(u^*)-\zeta_m(u^*)]>0$. But by Lemma 3.4, we get that $\lim_{J\to\infty}\sum_{|\alpha|=m}U_\alpha(x_0)\cdot[\zeta_m(u_J)-\zeta_m(u^*)]=0$. We have a contradiction. Hence we obtain (3.16).

Let the sequence $\{u_J\}_{J=1}^{\infty}$ satisfies (3.16) and all the lemmas above. Let $v\in\bigcup_{J=1}^{\infty}S_J$. Note that

$$\int_E|v||u_J|^{q(x)-1}\,dx\leqslant 2\big\||u_J|^{q(x)-1}\big\|_{q'(x),E}\|v\|_{q(x),E},$$

where E is a measurable subset of Ω. From (3.2), (3.4) and Theorem 2.8 we have $\{\||u_J|^{q(x)-1}\|_{q'(x)}\}_{J=1}^{\infty}$ is bounded. Using (f-2) and Theorem 2.4 we furthermore obtain that $\{f(x,u_J)v\}_{J=1}^{\infty}$ is absolutely equi-integrable. From (f-1) and (3.8) we have $\lim_{J\to\infty}f(x,u_J)v=f(x,u^*)v$ for a.e. $x\in\Omega$. By Vitali's Theorem we obtain

$$\lim_{J\to\infty}\int_\Omega f(x,u_J)v\,dx=\int_\Omega f\big(x,u^*\big)v\,dx.$$

Since

$$\int_E\big|A_\alpha\big(x,\xi_m(u_J)\big)\big|\big|D^\alpha v\big|\,dx\leqslant 2\big\|A_\alpha\big(x,\xi_m(u_J)\big)\big\|_{p'(x)}\big\|D^\alpha v\big\|_{p(x),E}$$

for $|\alpha|\leqslant m$, where E is a measurable subset of Ω, from (3.13) and Theorem 2.4 we see that $\{A_\alpha(x,\xi_m(u_J))D^\alpha v\}_{J=1}^{\infty}$ is absolutely equi-integrable for $|\alpha|\leqslant m$. It follows from (A-1), (3.8) and (3.16) that

$$\lim_{J\to\infty}A_\alpha\big(x,\xi_m(u_J)\big)D^\alpha v=A_\alpha\big(x,\xi_m\big(u^*\big)\big)D^\alpha v$$

for a.e. $x\in\Omega$ and $|\alpha|\leqslant m$. Hence from Vitali's Theorem we have

$$\lim_{J\to\infty}\int_\Omega A_\alpha\big(x,\xi_m(u_J)\big)D^\alpha v\,dx=\int_\Omega A_\alpha\big(x,\xi_m\big(u^*\big)\big)D^\alpha v\,dx$$

for $|\alpha|\leqslant m$ and $\lim_{J\to\infty}Q(u_J,v)=Q(u^*,v)$. By Lemma 3.2 we see that for any $v\in\bigcup_{J=1}^{\infty}S_J$, $\exists J\geqslant 1$ such that $v\in S_J$ and

$$Q(u_J,v)=G(v)+\int_\Omega f(x,u_J)v\,dx,$$

so

$$Q\big(u^*,v\big)=G(v)+\int_\Omega f\big(x,u^*\big)v\,dx\quad\text{for any } v\in\bigcup_{J=1}^{\infty}S_J.$$

It follows from Lemma 3.1(b) that $\forall v\in W_0^{m,p(x)}(\Omega)$, $\exists\{v_J\}_{J=1}^{\infty}$ with $v_J\in S_J$ such that $\lim_{J\to\infty}\|v_J-v\|_{m,p(x)}=0$. Hence by (A-2) and Theorem 2.1 we have

$$\int_\Omega\big|A_\alpha\big(x,\xi_m\big(u^*\big)\big)\big|\big|D^\alpha v_J-D^\alpha v\big|\,dx\leqslant C\big(\big\|h_0(x)\big\|_{p'(x)}+\big\||\xi_m|^{p(x)-1}\big\|_{p'(x)}\big)\big\|D^\alpha v_J-D^\alpha v\big\|_{p(x)}.$$

So $\lim_{J\to\infty}\int_\Omega|A_\alpha(x,\xi_m(u^*))||D^\alpha v_J-D^\alpha v|\,dx=0$. Then we have $\lim_{J\to\infty}Q(u^*,v_J)=Q(u^*,v)$. By (f-2), (3.2) and Theorem 2.1, $\lim_{J\to\infty}\int_\Omega f(x,u^*)v_J\,dx=\int_\Omega f(x,u^*)v\,dx$. By (3.1) $|G(v_J-v)|\leqslant C\|v_J-v\|_{m,p(x)}$, so $\lim_{J\to\infty}G(v_J)=G(v)$. It is immediate that (1.4) holds. The proof of Theorem 1.1 is completed. □

References

[1] E. Acerbi, G. Mingione, Regularity results for stationary electro-rheological fluids, Arch. Ration. Mech. Anal. 164 (2002) 213–259.
[2] E. Acerbi, G. Mingione, G.A. Seregin, Regularity results for parabolic systems related to a class of non-Newtonian fluids, Ann. Inst. H. Poincare Anal. Non Lineaire 21 (2004) 25–60.
[3] J. Chabrowski, Y. Fu, Existent of solutions for $p(x)$-Laplacian problems on a bounded domain, J. Math. Anal. Appl. 306 (2005) 604–618; Corrigendum: J. Math. Anal. Appl. 323 (2006) 1483.
[4] D. Edmunds, J. Lang, A. Nekvinda, On $L^{p(x)}$ norms, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 455 (1999) 219–225.
[5] D. Edmunds, J. Rákosník, Sobolev embedding with variable exponent, Studia Math. 143 (2000) 267–293.
[6] X. Fan, J. Shen, D. Zhao, Sobolev embedding theorems for spaces $W^{m,p(x)}$, J. Math. Anal. Appl. 262 (2001) 749–760.
[7] X. Fan, D. Zhao, On the spaces $L^{p(x)}$ and $W^{m,p(x)}$, J. Math. Anal. Appl. 263 (2001) 424–446.
[8] X. Fan, Q. Zhang, D. Zhao, Eigenvalues of $p(x)$-Laplacian Dirichlet problem, J. Math. Anal. Appl. 302 (2005) 306–317.
[9] M. Galewski, New variational method for $p(x)$-Laplacian equation, Bull. Austral. Math. Soc. 72 (2005) 53–65.
[10] O. Kováčik, J. Rákosník, On spaces $L^{p(x)}$ and $W^{m,p(x)}$, Czechoslovak Math. J. 41 (116) (1991) 592–618.
[11] B. Mihăilescu, V. Rădulescu, A multiplicity result for a nonlinear degenerate problem arising in the theory of electrorheological fluids, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 462 (2006) 2625–2641.
[12] L. Nirenberg, Topics in Nonlinear Functional Analysis, Lecture Notes, Courant Institute, New York, 1974.
[13] M. Rů žička, Electrorheological Fluids: Modeling and Mathematical Theory, Springer-Verlag, Berlin, 2000.
[14] V.L. Shapiro, Quasilinear ellipticity and the 1st eigenvalue, Comm. Partial Differential Equations 16 (11) (1991) 1819–1855.
[15] V. Zhikov, Averaging of functionals in the calculus of variations and elasticity, Math. USSR Izv. 29 (1987) 33–66.